Critical exponents for crossing probability on a torus

Alexi MORIN-DUCHESNE

Centre de Recherches Mathématiques
Université de Montréal
Advisor: Yvan Saint-Aubin

November 10, 2009

Outline

1. From spin models to crossing probabilities
   - Q-state Potts Model
   - FK Loop model
   - Crossing probabilities
2. Developments
   - The $\tau_r = 0$, $\tau_i \to \infty$ limit
   - The $\tau_r = \mu/\nu$, $\tau_i \to 0$ limit
   - $Q \to 0$

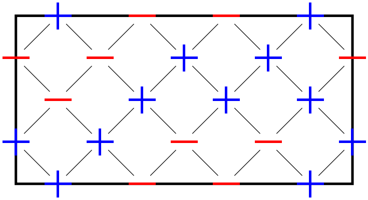
Part I

Introduction

The Potts Model (1952)

In the Q-Potts "spin" configuration, each spin can take one of Q values.

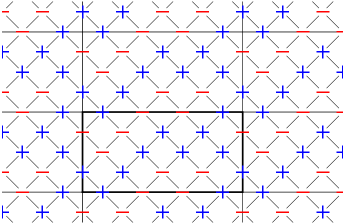
- Spins interact with nearest neighbors only
- The Energy of a configuration is given by $E(\sigma) = -J\sum_{\langle i,j\rangle}\delta_{i,j}$ where $\langle i,j\rangle$ indicates nearest neighbors
- Boltzmann distribution : each configuration has probability $\mathrm{Proba}(\sigma,\beta) = e^{-\beta E(\sigma)}/Z$
- $\beta = \frac{1}{kT}$ is inverse temperature
- Z is the Partition Function

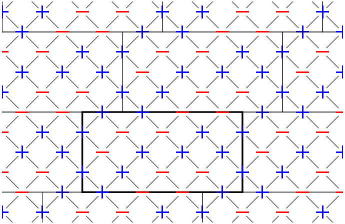


An example of spin configuration with $Q = 2$

Geometry and boundary conditions

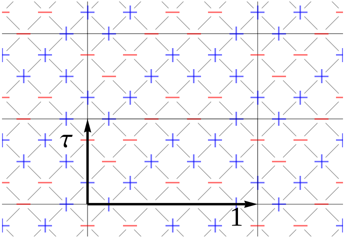
We'll be working with periodic boundary conditions, i.e. on the torus.

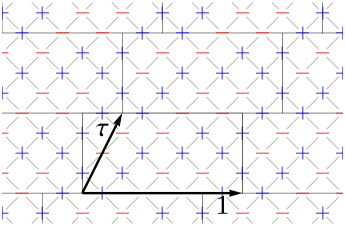




Geometry and boundary conditions

To describe the torus in the complex plane, we use two "periodicity vectors", set one to $1+0i$ and name the other one τ.

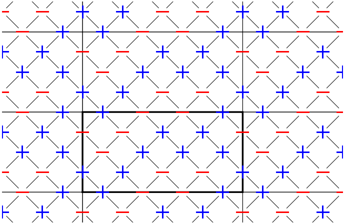


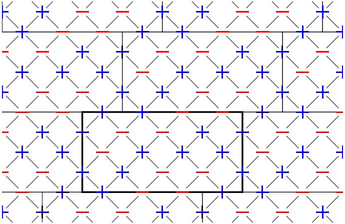


and will set $\tau = \tau_r + i\tau_i$

Phase transition

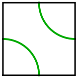
We'll be working with periodic boundary conditions, i.e. on the torus.

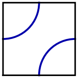


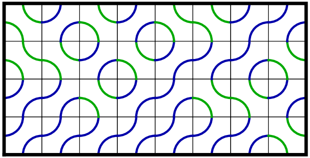


The Fortuin-Kasteleyn model (1972)

- A configuration in this model only consists of contours of spin domains.
- Building a configuration amounts to making one of these two choices for each tile:







A Fortuin-Kasteleyn configuration on the torus with $\tau = i/2$

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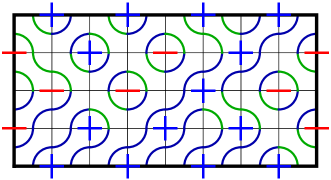
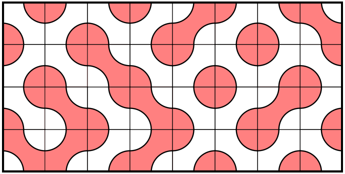


A Fortuin-Kasteleyn configuration on the torus with $\tau = i/2$

The Fortuin-Kasteleyn model (1972)

- $P(\sigma) = \frac{1}{Z} Q^{N_c} v^{N_l}$
- The Partition Function $Z = \sum_\sigma Q^{N_c} v^{N_l}$ is equal to Z in the Potts Model.
- $v = e^{J/k_B T} - 1$, N_c the # of connected components and N_l the # of links.
- Can be generalized to non-integer values for Q.
- At Critical Temperature, $v * Q^{-1/2} = 1$.



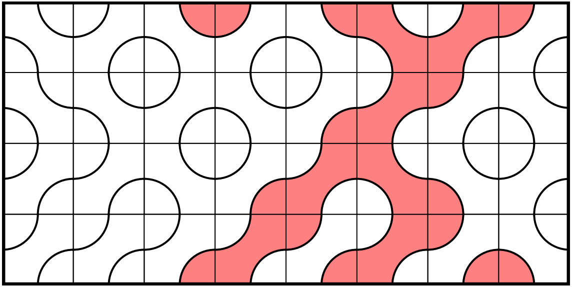
FK configuration, with $N_c = 7$ and $N_l = 9$ $(\tau = i/2)$.

Belongs to homotopy group $\{0\}$

FK clusters on a torus

FK configurations are classified in Homotopy groups:

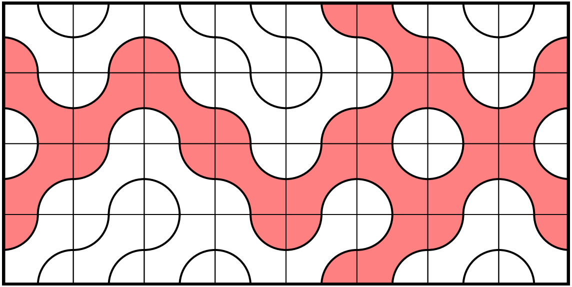
- $\{0\}$ (homotopic to a point)
- $\{a, b\}$ with $a \leftrightarrow$ and $b \updownarrow$ (along τ) .
- $\mathbb{Z} \times \mathbb{Z}$ (cross topology)



A $\{0, 1\}$ configuration ($\tau = i/2$)

FK clusters on a torus

In FK configurations, non-intersecting curves allows for one type of $\{a, b\}$ group only, and a and b must be coprime.



A $\mathbb{Z} \times \mathbb{Z}$ configuration ($\tau = i/2$)

$$Z = Z(\{0\}) + Z(\mathbb{Z} \times \mathbb{Z}) + \sum_{a \wedge b = 1} Z(\{a, b\})$$

We study $\pi(H) = Z(H)/Z$, for H a given homotopy group.

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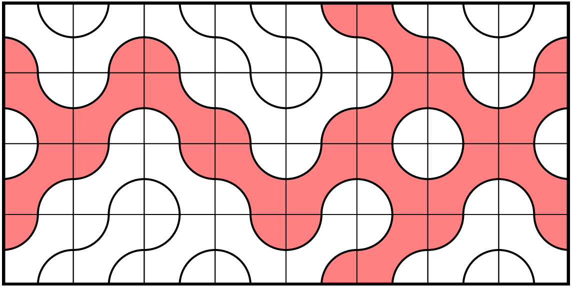

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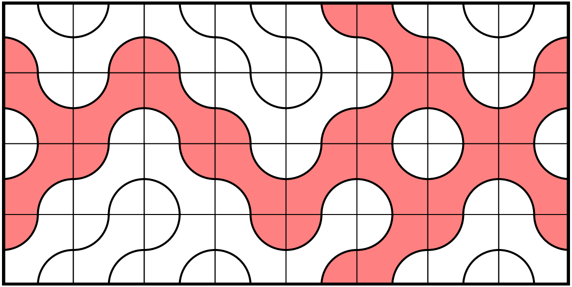

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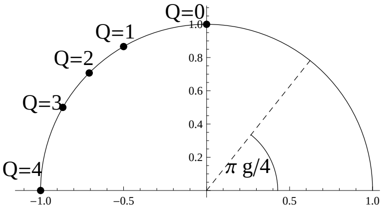
Partition functions

Work by B Nienhuis (1987), H Pinson (1994) and LP Arguin (2002) give us a first expression for $Z(\{0\})$, $Z(\mathbb{Z}\times\mathbb{Z})$ and $Z(\{a,b\})$:

- as lattice dimensions $\to \infty$
- for $Q\epsilon]0,4]$, as a function of new parameter $g(\epsilon]2,4])$: $Q = 4cos^2[\pi g/4]$.
- for any τ in the upper-half plane.



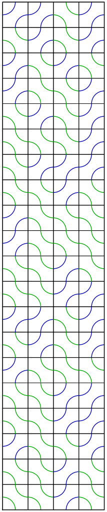
Part II

Developments

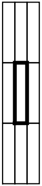
2 Developments

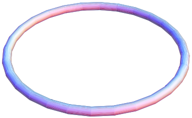
- The $\tau_r = 0$, $\tau_i \to \infty$ limit
- The $\tau_r = \mu/\nu$, $\tau_i \to 0$ limit
- $Q \to 0$

$\tau_r = 0,\ \tau_i \to \infty$: Objectives



- As torus $\to$ annulus, the relative length of $\{0,1\}$ curves goes to 0, and $\pi(\{1,0\}) \to 1$.





- Critical behavior should yield a power-law: $\pi(\{1,0\}) = 1 - \sum_{n,m} c_{n,m} q^{\gamma_n} \bar{q}^{\bar{\gamma}_m}$
- q is a variable depending only on the geometry of the torus, going to 0 as the limit is taken. The usual choice is $q = e^{2\pi i \tau}$.

$\tau_r = 0,\ \tau_i \to \infty$: Objectives

Conformal Field Theory gives a list of critical exponents in Kac table:

$$\Delta_{r,s} = \frac{[r - \frac{g}{4}s]^2 - (1 - \frac{g}{4})^2}{g}$$

for r,s positive integers.

6	$\frac{7}{2}$	$\frac{5}{3}$	$\frac{1}{2}$	0
$\frac{65}{16}$	$\frac{33}{16}$	$\frac{35}{48}$	$\frac{1}{16}$	$\frac{1}{16}$
$\frac{5}{2}$	1	$\frac{1}{6}$	0	$\frac{1}{2}$
$\frac{21}{16}$	$\frac{5}{16}$	$-\frac{1}{48}$	$\frac{5}{16}$	$\frac{21}{16}$
$\frac{1}{2}$	0	$\frac{1}{6}$	1	$\frac{5}{2}$
$\frac{1}{16}$	$\frac{1}{16}$	$\frac{35}{48}$	$\frac{33}{16}$	$\frac{65}{16}$
0	$\frac{1}{2}$	$\frac{5}{3}$	$\frac{7}{2}$	6

s

r

Kac table for Ising Model

$(Q = 2, g = 3)$

$\tau_r = 0,\ \tau_i \longrightarrow \infty$: Results for γ_n and $\overline{\gamma}_m$

For Potts model with integer Q, critical exponents for $\pi(\{0\})$, $\pi(\mathbb{Z}\times\mathbb{Z})$ and $\pi(\{1,0\})$ are in a Kac table extended to half-integer values for r and s!

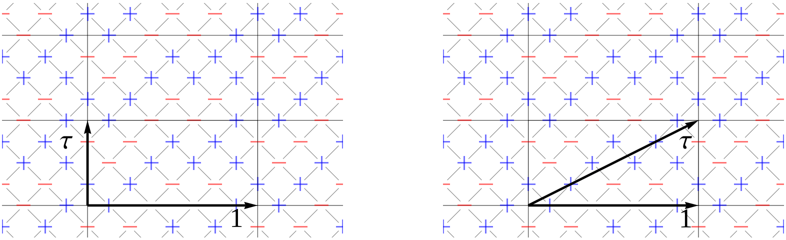
s

$\frac{5}{2}$	$\frac{5}{3}$	1	$\frac{1}{2}$	$\frac{1}{6}$	0	0
$\frac{119}{64}$	$\frac{221}{192}$	$\frac{39}{64}$	$\frac{15}{64}$	$\frac{5}{192}$	$-\frac{1}{64}$	$\frac{7}{64}$
$\frac{21}{16}$	$\frac{35}{48}$	$\frac{5}{16}$	$\frac{1}{16}$	$-\frac{1}{48}$	$\frac{1}{16}$	$\frac{5}{16}$
$\frac{55}{64}$	$\frac{77}{192}$	$\frac{7}{64}$	$-\frac{1}{64}$	$\frac{5}{192}$	$\frac{15}{64}$	$\frac{39}{64}$
$\frac{1}{2}$	$\frac{1}{6}$	0	0	$\frac{1}{6}$	$\frac{1}{2}$	1
$\frac{15}{64}$	$\frac{5}{192}$	$-\frac{1}{64}$	$\frac{7}{64}$	$\frac{77}{192}$	$\frac{55}{64}$	$\frac{95}{64}$
$\frac{1}{16}$	$-\frac{1}{48}$	$\frac{1}{16}$	$\frac{5}{16}$	$\frac{35}{48}$	$\frac{21}{16}$	$\frac{33}{16}$
$-\frac{1}{64}$	$\frac{5}{192}$	$\frac{15}{64}$	$\frac{39}{64}$	$\frac{221}{192}$	$\frac{119}{64}$	$\frac{175}{64}$
0	$\frac{1}{6}$	$\frac{1}{2}$	1	$\frac{5}{3}$	$\frac{5}{2}$	$\frac{7}{2}$

r

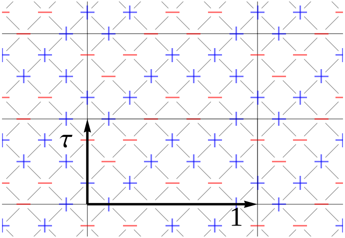
Kac table for Ising Model, with half integers.

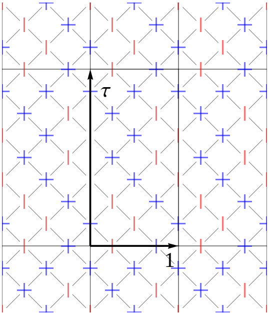
$\tau_r = \mu/\nu,\ \tau_i \to 0$: Modular transformations



$$\tau \to \tau + 1$$

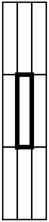
$\tau_r = \mu/\nu,\ \tau_i \longrightarrow 0$: Modular transformations

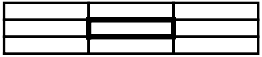




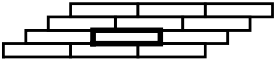
$$\tau \longrightarrow \frac{-1}{\tau}$$

$\tau_r = \mu/\nu,\ \tau_i \longrightarrow 0$: Results $(\mu \wedge \nu = 1)$





$\tau = \frac{i}{\kappa}.$



$\tau = \frac{\mu}{\nu} + \frac{i}{\nu^2\kappa}.$

$\tau = i\kappa.$

Are "equivalent" in the sense that, when, $\kappa \longrightarrow \infty$:

$$\pi(\{1,0\})_{\tau=i\kappa} = \pi(\{0,1\})_{\tau=\frac{i}{\kappa}} = \pi(\{\mu,\nu\})_{\tau=\frac{\mu}{\nu}+\frac{i}{\nu^2\kappa}}$$

$$\pi(\{0\})_{\tau=i\kappa} = \pi(\{0\})_{\tau=\frac{i}{\kappa}} = \pi(\{0\})_{\tau=\frac{\mu}{\nu}+\frac{i}{\nu^2\kappa}}$$

Dense polymers

What happens in the limit $Q \longrightarrow 0$?
We set $Q = \varepsilon$ and an expansion around $\varepsilon = 0$ gives:

- $\pi(\{0\}) \sim 1$
- $\pi(\mathbb{Z} \times \mathbb{Z}) \sim \varepsilon$
- $\pi(\{a, b\}) \sim \sqrt{\varepsilon}$
- $P(\sigma) = \frac{1}{Z} Q^{N_c} Q^{N_l/2}$

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Conclusions

- We were able to compute the homotopy probability for Fortuin-Kasteleyn clusters, as power laws in the geometric variable q, for two limiting cases. Numerical simulations support our results.
- We found the exponents to be in the integer and half-integer Kac table, at least for integer Q.
- We believe SLE methods should be able to find the same results we find for homotopy probability. The link between conformal loop models and the FK model has been shown to go as $\sqrt{Q} = -2\cos[4\pi/\kappa]$ with $\kappa \epsilon [4, 8]$.